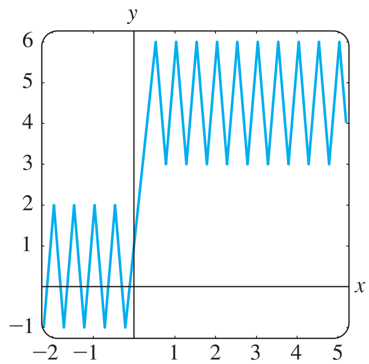

FIGURE 2.6.4 A not very helpful numerical solution curve

USING A NUMERICAL SOLVER Knowledge of the various numerical methods is not necessary in order to use a numerical solver. A solver usually requires that the differential equation be expressed in normal form $dy/dx = f(x, y)$. Numerical solvers that generate only curves usually require that you supply $f(x, y)$ and the initial data x_0 and y_0 and specify the desired numerical method. If the idea is to approximate the numerical value of $y(a)$, then a solver may additionally require that you state a value for h or, equivalently, give the number of steps that you want to take to get from $x = x_0$ to $x = a$. For example, if we wanted to approximate $y(4)$ for the IVP illustrated in Figure 2.6.3, then, starting at $x = 0$ it would take four steps to reach $x = 4$ with a step size of $h = 1$; 40 steps is equivalent to a step size of $h = 0.1$. Although we will not delve here into the many problems that one can encounter when attempting to approximate mathematical quantities, you should at least be aware of the fact that a numerical solver may break down near certain points or give an incomplete or misleading picture when applied to some first-order differential equations in the normal form. Figure 2.6.4 illustrates the graph obtained by applying Euler's method to a certain first-order initial-value problem $dy/dx = f(x, y)$, $y(0) = 1$. Equivalent results were obtained using three different commercial numerical solvers, yet the graph is hardly a plausible solution curve. (Why?) There are several avenues of recourse when a numerical solver has difficulties; three of the more obvious are decrease the step size, use another numerical method, and try a different numerical solver.

EXERCISES 2.6

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1 and 2 use Euler's method to obtain a four-decimal approximation of the indicated value. Carry out the recursion of (3) by hand, first using $h = 0.1$ and then using $h = 0.05$.

1. $y' = 2x - 3y + 1,\ y(1) = 5;\quad y(1.2)$

2. $y' = x + y^2,\ y(0) = 0;\quad y(0.2)$

In Problems 3 and 4 use Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$. Find an explicit solution for each initial-value problem and then construct tables similar to Tables 2.3 and 2.4.

3. $y' = y,\ y(0) = 1;\quad y(1.0)$

4. $y' = 2xy,\ y(1) = 1;\quad y(1.5)$

In Problems 5–10 use a numerical solver and Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$.

5. $y' = e^{-y},\ y(0) = 0;\quad y(0.5)$

6. $y' = x^2 + y^2,\ y(0) = 1;\quad y(0.5)$

7. $y' = (x - y)^2,\ y(0) = 0.5;\quad y(0.5)$

8. $y' = xy + \sqrt{y},\ y(0) = 1;\quad y(0.5)$

9. $y' = xy^2 - \dfrac{y}{x},\ y(1) = 1;\quad y(1.5)$

10. $y' = y - y^2,\ y(0) = 0.5;\quad y(0.5)$

In Problems 11 and 12 use a numerical solver to obtain a numerical solution curve for the given initial-value problem. First use Euler's method and then the RK4 method. Use $h = 0.25$ in each case. Superimpose both solution curves on the same coordinate axes. If possible, use a different color for each curve. Repeat, using $h = 0.1$ and $h = 0.05$.

11. $y' = 2(\cos x)y,\quad y(0) = 1$

12. $y' = y(10 - 2y),\quad y(0) = 1$

Discussion Problems

13. Use a numerical solver and Euler's method to approximate $y(1.0)$, where $y(x)$ is the solution to $y' = 2xy^2$, $y(0) = 1$. First use $h = 0.1$ and then use $h = 0.05$. Repeat, using the RK4 method. Discuss what might cause the approximations to $y(1.0)$ to differ so greatly.

Computer Lab Assignments

14. (a) Use a numerical solver and the RK4 method to graph the solution of the initial-value problem $y' = -2xy + 1,\ y(0) = 0$.

(b) Solve the initial-value problem by one of the analytic procedures developed earlier in this chapter.

(c) Use the analytic solution $y(x)$ found in part (b) and a CAS to find the coordinates of all relative extrema.

CHAPTER 2 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-3.

Answer Problems 1–4 without referring back to the text. Fill in the blanks or answer true or false.

1. The linear DE, $y' - ky = A$, where k and A are constants, is autonomous. The critical point __________ of the equation is a(n) __________ (attractor or repeller) for $k > 0$ and a(n) __________ (attractor or repeller) for $k < 0$.

2. The initial-value problem $x\dfrac{dy}{dx} - 4y = 0$, $y(0) = k$, has an infinite number of solutions for $k =$ __________ and no solution for $k =$ __________.

3. The linear DE, $y' + k_1y = k_2$, where k_1 and k_2 are nonzero constants, always possesses a constant solution. __________

4. The linear DE, $a_1(x)y' + a_2(x)y = 0$ is also separable. __________

In Problems 5 and 6 construct an autonomous first-order differential equation $dy/dx = f(y)$ whose phase portrait is consistent with the given figure.

5.

FIGURE 2.R.1 Graph for Problem 5

6.

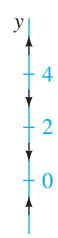

FIGURE 2.R.2 Graph for Problem 6

7. The number 0 is a critical point of the autonomous differential equation $dx/dt = x^n$, where n is a positive integer. For what values of n is 0 asymptotically stable? Semi-stable? Unstable? Repeat for the differential equation $dx/dt = -x^n$.

8. Consider the differential equation $dP/dt = f(P)$, where

$$f(P) = -0.5P^3 - 1.7P + 3.4.$$

The function $f(P)$ has one real zero, as shown in Figure 2.R.3. Without attempting to solve the differential equation, estimate the value of $\lim_{t\to\infty} P(t)$.

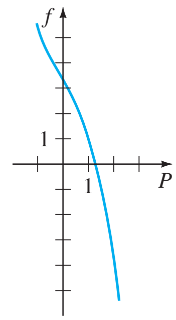

FIGURE 2.R.3 Graph for Problem 8

9. Figure 2.R.4 is a portion of a direction field of a differential equation $dy/dx = f(x, y)$. By hand, sketch two different solution curves—one that is tangent to the lineal element shown in black and one that is tangent to the lineal element shown in color.

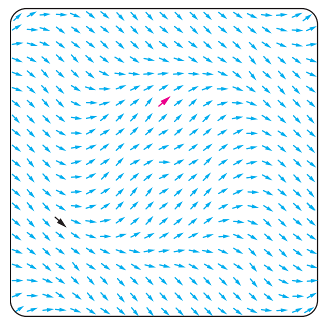

FIGURE 2.R.4 Portion of a direction field for Problem 9

10. Classify each differential equation as separable, exact, linear, homogeneous, or Bernoulli. Some equations may be more than one kind. Do not solve.

(a) $\dfrac{dy}{dx} = \dfrac{x - y}{x}$ **(b)** $\dfrac{dy}{dx} = \dfrac{1}{y - x}$

(c) $(x + 1)\dfrac{dy}{dx} = -y + 10$ **(d)** $\dfrac{dy}{dx} = \dfrac{1}{x(x - y)}$

(e) $\dfrac{dy}{dx} = \dfrac{y^2 + y}{x^2 + x}$ **(f)** $\dfrac{dy}{dx} = 5y + y^2$

(g) $y\,dx = (y - xy^2)\,dy$ **(h)** $x\dfrac{dy}{dx} = ye^{x/y} - x$

(i) $xy\,y' + y^2 = 2x$ **(j)** $2xy\,y' + y^2 = 2x^2$

(k) $y\,dx + x\,dy = 0$

(l) $\left(x^2 + \dfrac{2y}{x}\right)dx = (3 - \ln x^2)\,dy$

(m) $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x} + 1$ **(n)** $\dfrac{y}{x^2}\dfrac{dy}{dx} + e^{2x^3 + y^2} = 0$

In Problems 11–18 solve the given differential equation.

11. $(y^2 + 1)\,dx = y \sec^2 x\,dy$

12. $y(\ln x - \ln y)\,dx = (x \ln x - x \ln y - y)\,dy$

13. $(6x + 1)y^2 \dfrac{dy}{dx} + 3x^2 + 2y^3 = 0$

14. $\dfrac{dx}{dy} = -\dfrac{4y^2 + 6xy}{3y^2 + 2x}$

15. $t\dfrac{dQ}{dt} + Q = t^4 \ln t$

16. $(2x + y + 1)y' = 1$

17. $(x^2 + 4)\,dy = (2x - 8xy)\,dx$

18. $(2r^2 \cos\theta \sin\theta + r\cos\theta)\,d\theta + (4r + \sin\theta - 2r\cos^2\theta)\,dr = 0$

In Problems 19 and 20 solve the given initial-value problem and give the largest interval I on which the solution is defined.

19. $\sin x \dfrac{dy}{dx} + (\cos x)y = 0, \quad y\left(\dfrac{7\pi}{6}\right) = -2$

20. $\dfrac{dy}{dt} + 2(t + 1)y^2 = 0, \quad y(0) = -\frac{1}{8}$

21. **(a)** Without solving, explain why the initial-value problem

$$\frac{dy}{dx} = \sqrt{y}, \quad y(x_0) = y_0$$

has no solution for $y_0 < 0$.

(b) Solve the initial-value problem in part (a) for $y_0 > 0$ and find the largest interval I on which the solution is defined.

22. **(a)** Find an implicit solution of the initial-value problem

$$\frac{dy}{dx} = \frac{y^2 - x^2}{xy}, \quad y(1) = -\sqrt{2}.$$

(b) Find an explicit solution of the problem in part (a) and give the largest interval I over which the solution is defined. A graphing utility may be helpful here.

23. Graphs of some members of a family of solutions for a first-order differential equation $dy/dx = f(x, y)$ are shown in Figure 2.R.5. The graphs of two implicit solutions, one that passes through the point $(1, -1)$ and one that passes through $(-1, 3)$, are shown in red. Reproduce the figure on a piece of paper. With colored pencils trace out the solution curves for the solutions $y = y_1(x)$ and $y = y_2(x)$ defined by the implicit solutions such that $y_1(1) = -1$ and $y_2(-1) = 3$, respectively. Estimate the intervals on which the solutions $y = y_1(x)$ and $y = y_2(x)$ are defined.

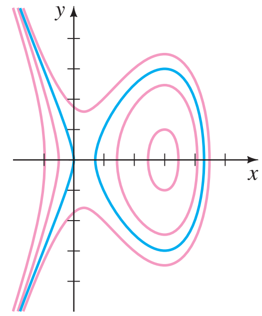

FIGURE 2.R.5 Graph for Problem 23

24. Use Euler's method with step size $h = 0.1$ to approximate $y(1.2)$, where $y(x)$ is a solution of the initial-value problem $y' = 1 + x\sqrt{y}$, $y(1) = 9$.

In Problems 25 and 26 each figure represents a portion of a direction field of an autonomous first-order differential equation $dy/dx = f(y)$. Reproduce the figure on a separate piece of paper and then complete the direction field over the grid. The points of the grid are (mh, nh), where $h = \frac{1}{2}$, m and n integers, $-7 \le m \le 7$, $-7 \le n \le 7$. In each direction field, sketch by hand an approximate solution curve that passes through each of the solid points shown in red. Discuss: Does it appear that the DE possesses critical points in the interval $-3.5 \le y \le 3.5$? If so, classify the critical points as asymptotically stable, unstable, or semi-stable.

25.

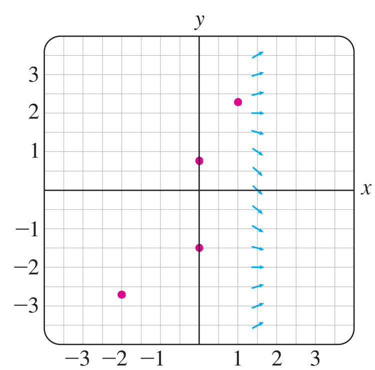

FIGURE 2.R.6 Portion of a direction field for Problem 25

26.

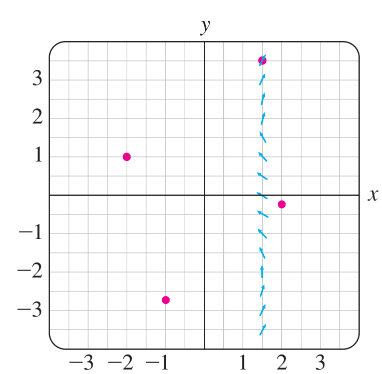

FIGURE 2.R.7 Portion of a direction field for Problem 26